A note on characterizing Hermitian curves via Baer sublines

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March 9, 2015

Abstract

We consider two families of point sets in (not necessarily finite) projective planes, one of which consists of the Hermitian curves, and give a common characterization of the point sets in both families. One of the properties we use to characterize them will be the existence of a certain configuration of Baer sublines.

Keywords: Hermitian curve, projective plane, Baer subline, unital
MSC2010: 51E20, 05B25

1 Introduction

A *unital* of the finite projective plane $\mathrm{PG}(2, q^2)$ (with q a prime power) is a set of $q^3 + 1$ points meeting each line in either 1 or $q + 1$ points. A line intersecting the unital in precisely $q + 1$ points is called a *secant line*, and such a secant line is called a *Baer secant line* if the intersection is a Baer subline. The standard examples of unitals are the Hermitian curves (also called *classical unitals*), and these have the property that all secant lines are Baer secant lines. Lefèvre-Percsy [7] and Faina-Korchmáros [6] proved that this property is sufficient to characterize Hermitian curves. They showed that every unital for which all secant lines are Baer secant lines necessarily is classical. A natural question which then arises is to ask how many Baer secant lines are really necessary to conclude that a unital must be classical. In the literature, one can find several results in this direction. The original results of [6, 7] were improved by Barwick [2]. The best result in the literature however seems to be the result which states that a unital $\mathcal{U}$ in $\mathrm{PG}(2, q^2)$ is classical as soon as there exists a point $x \in \mathcal{U}$ through which all secant lines are Baer secant lines and for which there exists one additional Baer secant line (not containing x). The proof of the latter result was established in the papers [3, 5, 8] and uses a deep result of Brown [4] regarding ovoids in the projective space $\mathrm{PG}(3, q)$. Important to mention is also the characterization result of Ball, Blokhuis and O'Keefe [1] which states that if q is a prime, then a unital in $\mathrm{PG}(2, q^2)$ is classical as soon as there are $(q^2 - 2)q$ Baer secant lines.

In the present paper, we give a characterization of Hermitian curves as certain sets of points that contain sufficiently many Baer secant lines (see condition (U1) below). We do no longer require in advance that the set is a unital as the earlier-mentioned characterization results do, but instead we require another condition (see condition (U3) below). The obtained characterization result will moreover be valid in the infinite case. In fact, the properties (U1), (U2) and (U3) below not only allow to characterize Hermitian curves but also the members of another family of point sets.

2 The main result

Let V be a 3-dimensional vector space over a field $\mathbb{F}$, and denote by $\mathrm{PG}(V)$ the projective plane associated with V.

Suppose X is a set of points of $\mathrm{PG}(V)$. A line L of $\mathrm{PG}(V)$ is called an *exterior line* if $L \cap X = \emptyset$, a *tangent line* if $|L \cap X| = 1$ and a *secant line* if $|L \cap X| \geq 2$.

Suppose $\mathbb{K}$ is a subfield of index 2 of $\mathbb{F}$. If $\bar{v}_1$ and $\bar{v}_2$ are two linearly independent vectors of V, then the set of all points of the form $\langle \lambda_1 \bar{v}_1 + \lambda_2 \bar{v}_2 \rangle$, $(\lambda_1, \lambda_2) \in (\mathbb{K} \times \mathbb{K}) \setminus \{(0,0)\}$ is called a *Baer-*$\mathbb{K}$*-subline* of $\mathrm{PG}(V)$. If $\bar{v}_1$, $\bar{v}_2$ and $\bar{v}_3$ are three linearly independent vectors of V, then the set of all points of the form $\langle \lambda_1 \bar{v}_1 + \lambda_2 \bar{v}_2 + \lambda_3 \bar{v}_3 \rangle$, $(\lambda_1, \lambda_2, \lambda_3) \in (\mathbb{K} \times \mathbb{K} \times \mathbb{K}) \setminus \{(0,0,0)\}$ is called a *Baer-*$\mathbb{K}$*-subplane* of $\mathrm{PG}(V)$.

Suppose $\mathbb{K}$ is a subfield of $\mathbb{F}$ such that $\mathbb{F}$ is a separable quadratic extension of $\mathbb{K}$, and denote by ψ the unique nontrivial automorphism of $\mathbb{F}$ fixing $\mathbb{K}$ elementwise. We call a set of points of $\mathrm{PG}(V)$ a $\mathbb{K}$*-Hermitian curve* if it has equation $X_0^{\psi+1} + X_1 X_2^{\psi} + X_2 X_1^{\psi} = 0$ with respect to a certain reference system. Every point of a $\mathbb{K}$-Hermitian curve is contained in a unique tangent line. If $\mathcal{U}$ is a $\mathbb{K}$-Hermitian curve and L a line of $\mathrm{PG}(V)$, then $L \cap \mathcal{U}$ is either empty, a singleton or a Baer-$\mathbb{K}$-subline. Moreover, if $L \cap \mathcal{U}$ is a Baer-$\mathbb{K}$-subline, then there exists a unique point l such that lx is a tangent line for every $x \in L \cap \mathcal{U}$. In fact, $l = L^{\beta}$ where β is the unitary polarity associated with $\mathcal{U}$.

Suppose again that $\mathbb{K}$ is a subfield of index 2 of $\mathbb{F}$. If X is a set of points of $\mathrm{PG}(V)$ and x_1, x_2 are two distinct points of X, then we say that X satisfies *Property (∗) with respect to* $(\mathbb{K}, x_1, x_2)$ if the following conditions are satisfied:

(U1) Any secant line through x_1 or x_2 intersects X in a Baer-$\mathbb{K}$-subline.

(U2) $X \setminus x_1 x_2 \neq \emptyset$.

(U3) There exists a point $l \in \mathrm{PG}(V) \setminus x_1 x_2$ such that lx is a tangent line for every point $x \in x_1 x_2 \cap X$.

We say that X satisfies *Property (∗) with respect to* $\mathbb{K}$ if there exist two distinct points $x_1, x_2 \in X$ such that X satisfies Property (∗) with respect to $(\mathbb{K}, x_1, x_2)$. We will prove the following result.

Theorem 2.1 (1) *If* $\mathbb{F}$ *is a separable quadratic extension of the field* $\mathbb{K}$*, then the sets of points of* $\mathrm{PG}(V)$ *satisfying Property (∗) with respect to* $\mathbb{K}$ *are precisely the* $\mathbb{K}$*-Hermitian curves of* $\mathrm{PG}(V)$.

(2) *If $\mathbb{F}$ is an inseparable quadratic extension of the field $\mathbb{K}$, then the sets of points of* $\mathrm{PG}(V)$ *satisfying Property (∗) with respect to $\mathbb{K}$ are precisely the sets of points described by a condition of the form "$\lambda X_0^2 + X_1X_2 \in \mathbb{K}$", where $\lambda \in \mathbb{F} \setminus \mathbb{K}$ and (X_0, X_1, X_2) denote the homogeneous coordinates with respect to a fixed reference system.*

Note that if $\mathbb{F}$ is an inseparable quadratic extension of $\mathbb{K}$, then the characteristic of $\mathbb{F}$ is 2 and $\mathbb{F}^2 := \{x^2 \,|\, x \in \mathbb{F}\} \subseteq \mathbb{K}$. This implies that the condition $\lambda X_0^2 + X_1X_2 \in \mathbb{K}$ is well-defined: if (X_0, X_1, X_2) satisfies this condition, then also $(\lambda X_0, \lambda X_1, \lambda X_2)$ with $\lambda \in \mathbb{F}^*$ satisfies the condition.

In Theorem 2.1, we characterized two families of sets (among which the $\mathbb{K}$-Hermitian curves) as sets of points satisfying the three properties (U1), (U2) and (U3). With the aid of a few examples, we now show that each of these conditions is in some sense necessary.

Examples. (a) Suppose X is a Baer-$\mathbb{K}$-subline, x_1, x_2 are two distinct points of X and l is any point of $\mathrm{PG}(V) \setminus x_1x_2$. Then X satisfies Properties (U1) and (U3), but not Property (U2).

(b) Suppose X is a Baer-$\mathbb{K}$-subplane of $\mathrm{PG}(V)$ and x_1, x_2 are two distinct points of X. Then X satisfies the Properties (U1) and (U2), but not (U3). Indeed, every point of $\mathrm{PG}(V) \setminus X$ is then contained in a unique secant line.

(c) Let $\mathcal{L}$ denote a collection of lines through a point l and L a line not containing l such that $\{x \in L \,|\, xl \in \mathcal{L}\}$ is a Baer-$\mathbb{K}$-subline L'. Let $x_1 \in L'$, let L'' denote a Baer-$\mathbb{K}$-subline of L such that $L' \cap L'' = \{x_1\}$ and let $x_2 \in L'' \setminus L'$. Let Y denote the set of points of $\mathrm{PG}(V)$ contained on a line of $\mathcal{L}$, and put $X := (Y \setminus (lx_1 \cup L')) \cup L''$. Then X satisfies Properties (U2) and (U3), but not Property (U1) if $|\mathbb{F}| > 4$. However, it is still true that every secant line through x_1 intersects X in a Baer-$\mathbb{K}$-subline (as well as one line through x_2, namely x_1x_2).

3 $(\mathbb{F}, \mathbb{K})$-sets

Proposition 3.1 *Suppose $\mathbb{F}$ is a separable quadratic extension of the field $\mathbb{K}$, and let ψ denote the unique nontrivial element of* $\mathrm{Gal}(\mathbb{F}/\mathbb{K})$. *Let $a_0, a_1, a_2 \in \mathbb{F}^*$ such that $\frac{a_1}{a_2} \notin \mathbb{K}$. Then the $\mathbb{K}$-Hermitian curve $\mathcal{U}$ of* $\mathrm{PG}(V)$ *having equation*

$$\frac{a_2a_1^\psi - a_1a_2^\psi}{a_0^{\psi+1}} X_0^{\psi+1} + X_1X_2^\psi - X_2X_1^\psi = 0$$

with respect to some ordered basis $(\bar{e}_0, \bar{e}_1, \bar{e}_2)$ of V consists of all points of the form

$$\langle \lambda_1\bar{e}_1 + \lambda_2\bar{e}_2 \rangle,$$

where $(\lambda_1, \lambda_2) \in (\mathbb{K} \times \mathbb{K}) \setminus \{(0, 0)\}$, and all points of the form

$$\langle a_0\bar{e}_0 + (\lambda_1a_1 + \lambda_2a_2)\bar{e}_1 + (\mu_1a_1 + \mu_2a_2)\bar{e}_2 \rangle,$$

where $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{K}$ *with* $\begin{vmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{vmatrix} = 1$.

Proof. Let $p = (X_0, X_1, X_2)$ be a point of $\mathrm{PG}(V)$.

If $X_0 = 0$, then $p \in \mathcal{U}$ if and only if $X_1 X_2^\psi - X_2 X_1^\psi = 0$, i.e. if and only if p is of the form $\langle \lambda_1 \bar{e}_1 + \lambda_2 \bar{e}_2 \rangle$, where $(\lambda_1, \lambda_2) \in (\mathbb{K} \times \mathbb{K}) \setminus \{(0,0)\}$.

If $X_0 \neq 0$, then we may suppose that $X_0 = a_0$. Since a_1, a_2 are linearly independent over $\mathbb{K}$, we have $X_1 = \lambda_1 a_1 + \lambda_2 a_2$ and $X_2 = \mu_1 a_1 + \mu_2 a_2$ for certain $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{K}$. The point p belongs to $\mathcal{U}$ if and only if

$$-(a_1 a_2^\psi - a_2 a_1^\psi) + (\lambda_1 a_1 + \lambda_2 a_2)(\mu_1 a_1^\psi + \mu_2 a_2^\psi) - (\mu_1 a_1 + \mu_2 a_2)(\lambda_1 a_1^\psi + \lambda_2 a_2^\psi)$$

$$= (\lambda_1 \mu_2 - \mu_1 \lambda_2 - 1) \cdot (a_1 a_2^\psi - a_2 a_1^\psi) = 0.$$

Since $a_1 a_2^\psi - a_2 a_1^\psi \neq 0$, this is equivalent with demanding that $\begin{vmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{vmatrix} = 1$. ■

Remark. The $\mathbb{K}$-Hermitian curves of $\mathrm{PG}(V)$ are also the sets of points described by equations of the form $\lambda X_0^{\psi+1} + X_1 X_2^\psi - X_2 X_1^\psi = 0$, where $\lambda \in \mathbb{F}^*$ such that $\lambda^\psi = -\lambda$. If $a_0, a_1, a_2 \in \mathbb{F}^*$ with $\frac{a_1}{a_2} \notin \mathbb{K}$, then $\lambda := \frac{a_2 a_1^\psi - a_1 a_2^\psi}{a_0^{\psi+1}}$ satisfies $\lambda \neq 0$ and $\lambda^\psi = -\lambda$. Conversely, every $\lambda \in \mathbb{F}^*$ for which $\lambda^\psi = -\lambda$ is of the form $\frac{a_2 a_1^\psi - a_1 a_2^\psi}{a_0^{\psi+1}}$ for certain $a_0, a_1, a_2 \in \mathbb{F}^*$ with $\frac{a_1}{a_2} \notin \mathbb{K}$. Indeed, take $a_0 = a_1 = 1$ and $a_2 = \frac{\lambda\eta}{\eta - \eta^\psi}$ where $\eta \in \mathbb{F} \setminus \mathbb{K}$.

Proposition 3.2 *Suppose* $\mathbb{F}$ *is an inseparable quadratic extension of the field* $\mathbb{K}$. *Let* $a_0, a_1, a_2 \in \mathbb{F}^*$ *such that* $\frac{a_1}{a_2} \notin \mathbb{K}$. *Then the set* X *of points described by the condition*

$$\frac{a_1 a_2}{a_0^2} X_0^2 + X_1 X_2 \in \mathbb{K},$$

where (X_0, X_1, X_2) *denote the homogeneous coordinates with respect to some ordered basis* $(\bar{e}_0, \bar{e}_1, \bar{e}_2)$ *of* V, *consists of all points of the form*

$$\langle \lambda_1 \bar{e}_1 + \lambda_2 \bar{e}_2 \rangle,$$

where $(\lambda_1, \lambda_2) \in (\mathbb{K} \times \mathbb{K}) \setminus \{(0,0)\}$, *and all points of the form*

$$\langle a_0 \bar{e}_0 + (\lambda_1 a_1 + \lambda_2 a_2)\bar{e}_1 + (\mu_1 a_1 + \mu_2 a_2)\bar{e}_2 \rangle,$$

where $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{K}$ *with* $\begin{vmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{vmatrix} = 1$.

Proof. Let $p = (X_0, X_1, X_2)$ be an arbitrary point of $\mathrm{PG}(V)$.

If $X_0 = 0$, then $p \in X$ if and only if $X_1 X_2 \in \mathbb{K}$. If $X_2 \neq 0$, then since $\mathbb{F}^2 \subseteq \mathbb{K}$, the condition $X_1 X_2 \in \mathbb{K}$ is equivalent with $\frac{X_1}{X_2} \in \mathbb{K}$. So, if $X_0 = 0$, then $p \in X$ if and only if p is of the form $\langle \lambda_1 \bar{e}_1 + \lambda_2 \bar{e}_2 \rangle$, where $(\lambda_1, \lambda_2) \in (\mathbb{K} \times \mathbb{K}) \setminus \{(0,0)\}$.

If $X_0 \neq 0$, then we may suppose that $X_0 = a_0$. Since a_1, a_2 are linearly independent over $\mathbb{K}$, we have $X_1 = \lambda_1 a_1 + \lambda_2 a_2$ and $X_2 = \mu_1 a_1 + \mu_2 a_2$ for certain $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{K}$. The point p belongs to X if and only if

$$a_1 a_2 + (\lambda_1 a_1 + \lambda_2 a_2)(\mu_1 a_1 + \mu_2 a_2)$$

$$= (\lambda_1 \mu_1 a_1^2 + \lambda_2 \mu_2 a_2^2) + a_1 a_2 (1 + \lambda_1 \mu_2 + \lambda_2 \mu_1) \in \mathbb{K}.$$

Since $a_1^2, a_2^2 \in \mathbb{K}$, we have $\lambda_1 \mu_1 a_1^2 + \lambda_2 \mu_2 a_2^2 \in \mathbb{K}$, and since $\frac{a_1}{a_2} \notin \mathbb{K}$, we also have $a_1 a_2 \notin \mathbb{K}$. So, the point p belongs to X if and only if $1 + \lambda_1 \mu_2 + \lambda_2 \mu_1 = 0$, i.e. if and only if

$$\begin{vmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{vmatrix} = 1.$$

■

Remark. Suppose $\mathbb{F}$ is an inseparable quadratic extension of the field $\mathbb{K}$. If $a_0, a_1, a_2 \in \mathbb{F}^*$ such that $\frac{a_1}{a_2} \notin \mathbb{K}$, then $a_0^2 \in \mathbb{K}^*$ and $a_1 a_2 \notin \mathbb{K}^*$ and hence $\frac{a_1 a_2}{a_0^2} \notin \mathbb{K}$. Conversely, if $\lambda \in \mathbb{F} \setminus \mathbb{K}$, then $\lambda = \frac{a_1 a_2}{a_0^2}$ for certain $a_0, a_1, a_2 \in \mathbb{F}^*$ such that $\frac{a_1}{a_2} \notin \mathbb{K}$. Indeed, we can take $a_0 = 1$, $a_1 = 1$ and $a_2 = \lambda$.

We introduce a common terminology for the sets of points occurring in Propositions 3.1 and 3.2. Suppose $\mathbb{K}$ is a subfield of index 2 of $\mathbb{F}$. A set X of points of $\mathrm{PG}(V)$ is called an $(\mathbb{F}, \mathbb{K})$-*set with respect to some line* L of $\mathrm{PG}(V)$ if there exist $a_0, a_1, a_2 \in \mathbb{F}^*$ with $\frac{a_1}{a_2} \notin \mathbb{K}$ and an ordered basis $(\bar{e}_0, \bar{e}_1, \bar{e}_2)$ of V such that $L = \langle \bar{e}_1, \bar{e}_2 \rangle$ and X consists of all points of the form $\langle \lambda_1 \bar{e}_1 + \lambda_2 \bar{e}_2 \rangle$, where $(\lambda_1, \lambda_2) \in (\mathbb{K} \times \mathbb{K}) \setminus \{(0,0)\}$, and all points of the form $\langle a_0 \bar{e}_0 + (\lambda_1 a_1 + \lambda_2 a_2)\bar{e}_1 + (\mu_1 a_1 + \mu_2 a_2)\bar{e}_2 \rangle$, where $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{K}$ with $\begin{vmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{vmatrix} = 1$. So, we have the following.

Corollary 3.3 (*a*) *If* $\mathbb{F}$ *is a separable quadratic extension of the field* $\mathbb{K}$*, then the* $(\mathbb{F}, \mathbb{K})$-*sets are precisely the* $\mathbb{K}$-*Hermitian curves of* $\mathrm{PG}(V)$.

(*b*) *If* $\mathbb{F}$ *is an inseparable quadratic extension of the field* $\mathbb{K}$*, then the* $(\mathbb{F}, \mathbb{K})$-*sets are precisely the sets of points described by a condition of the form "*$\lambda X_0^2 + X_1 X_2 \in \mathbb{K}$*", where* $\lambda \in \mathbb{F} \setminus \mathbb{K}$ *and* (X_0, X_1, X_2) *denote the homogeneous coordinates of the points with respect to a fixed reference system.*

We prove the following property of $(\mathbb{F}, \mathbb{K})$-sets.

Proposition 3.4 *Suppose* $\mathbb{F}$ *is a quadratic extension of the field* $\mathbb{K}$*. If* X *is an* $(\mathbb{F}, \mathbb{K})$-*set and* x_1, x_2 *are two distinct points of* X*, then* X *satisfies Property* $(*)$ *with respect to* $(\mathbb{K}, x_1, x_2)$*. Moreover, every point of* X *is contained in a unique tangent line.*

Proof. If $\mathbb{F}$ is a separable quadratic extension of $\mathbb{K}$, then X is a $\mathbb{K}$-Hermitian curve, and we already know then that X must satisfy Property $(\mathbb{K}, x_1, x_2)$ and that every point of X is contained in a unique tangent line.

Suppose therefore that $\mathbb{F}$ is an inseparable quadratic extension of $\mathbb{K}$. Then take an ordered basis $(\bar{e}_0, \bar{e}_1, \bar{e}_2)$ of V and a $\lambda \in \mathbb{F} \setminus \mathbb{K}$ such that X is described by the condition

$$\lambda X_0^2 + X_1 X_2 \in \mathbb{K}.$$

Put $l := \langle \bar{e}_0 \rangle$. Then $l \notin X$ since $\lambda \notin \mathbb{K}$. To prove the proposition, it suffices to show the following:

> (†) For every point $x \in X$, the line lx is a tangent line and every other line through x intersects X in a Baer-$\mathbb{K}$-subline.

Now, consider the quadratic form $q : V \to \mathbb{F}$ defined by

$$q(X_0\bar{e}_0 + X_1\bar{e}_1 + X_2\bar{e}_2) = \lambda X_0^2 + X_1X_2,$$

and denote by $f : V \times V \to \mathbb{F}$ the associated bilinear form, i.e. $f(\bar{v}, \bar{w}) = q(\bar{v} + \bar{w}) - q(\bar{v}) - q(\bar{w})$, $\forall \bar{v}, \bar{w} \in V$. Then f is a degenerate alternating bilinear form with radical $\langle \bar{e}_0 \rangle$.

We show the validity of (†). Let $\bar{v} \in V$ such that $x = \langle \bar{v} \rangle$ and let L be a line through x. Then L is determined by a 2-space $\langle \bar{v}, \bar{w} \rangle$ of V. Every point of $L \setminus \{x\}$ has the form $\langle \mu\bar{v} + \bar{w} \rangle$ where $\mu \in \mathbb{F}$. Note that $q(\mu\bar{v} + \bar{w}) = \mu^2 \cdot q(\bar{v}) + \mu \cdot f(\bar{v}, \bar{w}) + q(\bar{w})$ and $\mu^2 q(\bar{v}) \in \mathbb{K}$.

If $f(\bar{v}, \bar{w}) \neq 0$, then $\langle \mu\bar{v} + \bar{w} \rangle \in X$ if and only if μ is of the form $\frac{k - q(\bar{w})}{f(\bar{v}, \bar{w})}$ where $k \in \mathbb{K}$, and it is straightforward to verify that for such values of μ, the points $\langle \bar{v} \rangle$, $\langle \mu\bar{v} + \bar{w} \rangle$ form a Baer-$\mathbb{K}$-subline of L.

If $f(\bar{v}, \bar{w}) = 0$, then $\langle \bar{v}, \bar{w} \rangle = \langle \bar{v}, \bar{e}_0 \rangle$ and without loss of generality, we may suppose that $\bar{w} = \bar{e}_0$. Then $q(\mu\bar{v} + \bar{w}) = \mu^2 \cdot q(\bar{v}) + \lambda \notin \mathbb{K}$ for all $\mu \in \mathbb{K}$. In this case, L is a tangent line. ∎

4 Proof of Theorem 2.1

In this section, we suppose that $\mathbb{F}$ is a quadratic extension of the field $\mathbb{K}$. We also suppose that X is a set of points of $\mathrm{PG}(V)$ satisfying Property $(*)$ with respect to $(\mathbb{K}, x_1, x_2)$, where x_1 and x_2 are two distinct points of X. Put $L := x_1x_2$. Then $L \cap X$ is a Baer-$\mathbb{K}$-subline of L. Let l be a point of $\mathrm{PG}(V) \setminus L$ such that lx is a tangent line for every point $x \in L \cap X$. We choose an ordered basis $(\bar{e}_0, \bar{e}_1, \bar{e}_2)$ of V such that $l = \langle \bar{e}_0 \rangle$, $x_1 = \langle \bar{e}_1 \rangle$, $x_2 = \langle \bar{e}_2 \rangle$ and $L \cap X$ consists of all points of the form $\langle \lambda_1\bar{e}_1 + \lambda_2\bar{e}_2 \rangle$, where $(\lambda_1, \lambda_2) \in (\mathbb{K} \times \mathbb{K}) \setminus \{(0, 0)\}$.

Lemma 4.1 *If $X \setminus L$ contains a point $y = \langle a_0\bar{e}_0 + a_1\bar{e}_1 + a_2\bar{e}_2 \rangle$ with $a_0, a_1, a_2 \in \mathbb{F}$, then a_1, a_2 are linearly independent over $\mathbb{K}$. Hence, $a_0, a_1, a_2 \notin \mathbb{F}^*$ and $\frac{a_1}{a_2} \notin \mathbb{K}$.*

Proof. Since $y \neq l$, we have $(a_1, a_2) \neq (0, 0)$. If a_1, a_2 were linearly dependent over $\mathbb{K}$, then $x = \langle a_1\bar{e}_1 + a_2\bar{e}_2 \rangle$ were a point of $L \cap X$ and $y \in lx$, an obvious contradiction. ∎

Lemma 4.2 *Suppose $X \backslash L$ contains the point $y = \langle a_0\bar{e}_0 + a_1\bar{e}_1 + a_2\bar{e}_2\rangle$. Then for all $\lambda \in \mathbb{K}$, the set X also contains the points $\langle a_0\bar{e}_0 + (a_1 + \lambda a_2)\bar{e}_1 + a_2\bar{e}_2\rangle$ and $\langle a_0\bar{e}_0 + a_1\bar{e}_1 + (a_2 + \lambda a_1)\bar{e}_2\rangle$.*

Proof. Consider the unique Baer-$\mathbb{K}$-subplane B containing the Baer-$\mathbb{K}$-sublines $x_1y \cap X$ and $x_1x_2 \cap X$. Through the point l, there exists a line M such that $M \cap B$ is a Baer-$\mathbb{K}$-subline. As M meets $L \cap X$, it is a tangent line. As $M \cap x_1y$ and $M \cap x_1x_2$ define two points of X contained in this tangent line, we have $M \cap x_1y = M \cap x_1x_2$, i.e. $M = lx_1$. So, the unique intersection point $\langle a_0\bar{e}_0 + a_1\bar{e}_1\rangle$ of the lines lx_1 and yx_2 must belong to B. Hence, the unique intersection point $\langle a_0\bar{e}_0 + (a_1 + \lambda a_2)\bar{e}_1 + a_2\bar{e}_2\rangle$ of the line x_1y with the unique line through $\langle a_0\bar{e}_0 + a_1\bar{e}_1\rangle$ and $\langle \lambda\bar{e}_1 + \bar{e}_2\rangle$ is contained in B and hence also in X.

By symmetry, we should also have that $\langle a_0\bar{e}_0 + a_1\bar{e}_1 + (a_2 + \lambda a_1)\bar{e}_2\rangle$ belongs to $X \setminus L$.

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By successive application of Lemma 4.2, we find:

Corollary 4.3 *Suppose $X \setminus L$ contains the point $y = \langle a_0\bar{e}_0 + a_1\bar{e}_1 + a_2\bar{e}_2\rangle$. Then $X \setminus L$ contains all points of the form*

$$\langle a_0\bar{e}_0 + (\lambda_1 a_1 + \lambda_2 a_2)\bar{e}_1 + (\mu_1 a_1 + \mu_2 a_2)\bar{e}_2\rangle,$$

where $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{K}$ with $\begin{vmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{vmatrix} = 1$.

Now, let Y denote the $(\mathbb{F}, \mathbb{K})$-set consisting of all points of the form $\langle \lambda_1\bar{e}_1 + \lambda_2\bar{e}_2\rangle$, where $(\lambda_1, \lambda_2) \in (\mathbb{K} \times \mathbb{K}) \setminus \{(0, 0)\}$, and all points of the form $\langle a_0\bar{e}_0 + (\lambda_1 a_1 + \lambda_2 a_2)\bar{e}_1 + (\mu_1 a_1 + \mu_2 a_2)\bar{e}_2\rangle$, where $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{K}$ with $\begin{vmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{vmatrix} = 1$. Then $Y \subseteq X$ by Corollary 4.3.

The following proposition, in combination with Corollary 3.3 and Proposition 3.4, finishes the proof of Theorem 2.1.

Proposition 4.4 *The set X coincides with the $(\mathbb{F}, \mathbb{K})$-set Y.*

Proof. Since $Y \subseteq X$, the unique line through x_1 tangent to Y necessarily coincides with lx_1 and is also the unique line through x_1 tangent to X. Suppose Y is properly contained in X, and denote by x an arbitrary point of $X \setminus Y$. Then xx_1 is not tangent to X and hence also not to Y. Now, $X \cap x_1x$ and $Y \cap x_1x$ are two Baer-$\mathbb{K}$-sublines of x_1x and $Y \cap x_1x$ would be properly contained in $X \cap x_1x$ (since $x \in (X \cap x_1x) \setminus (Y \cap x_1x)$). This is clearly impossible. ■

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